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ON CONTACT PROBLEMS IN AN INHOMOGENEOUS HALF-SPACE

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Abstract—Asymmetric contact problems are considered for an inhomogeneous half-space whose elastic modulus is a power function of the depth and where the contact region is assumed to be a circle. Exact solution of the two-dimensional integral equation involved is obtained in a closed form by means of a special integral representation of the kernel. An expression is derived for the displacements outside the contact region directly in terms of the known displacements inside the region. The case of arbitrary loading outside of a punch is discussed as an illustrative example.

1. INTRODUCTION

This paper considers a nonhomogeneous half-space $z \geq 0$ whose elastic modulus is a power function of the depth in the form, $E = E_0 z^\alpha$, where $E_0 = \text{constant}$, $|\alpha| < 1$, and the Poisson's ratio ν is assumed to be constant. Several types of boundary value problems have been solved recently by different authors. The action of a concentrated force on the boundary of the half-space was considered by Rostovtsev and Khranevskaya[1] and by Plevako[2]. Certain axisymmetric problems were solved by Puro[3] and by Popov[4]. The solution of the asymmetric contact problem in a series form was first given by Rostovtsev[5]. An exact solution of the same problem in a closed form is presented in this paper employing a special integral representation of the kernel of the governing integral equation. The different properties of the exact analytical solution obtained are investigated in a later section of this paper. An illustrative example of an external crack in the elastic space under arbitrary pressure applied in opposing directions of the sides of the crack is considered also.

2. FORMULATION OF THE PROBLEM AND SOLUTION

Consider the following mixed boundary-value problem for the above mentioned nonhomogeneous half-space. Inside a circle $\rho = a$ certain arbitrary normal displacements $w(\rho, \phi)$ are prescribed, while the boundary $z = 0$ of the half-space outside the circle is free of stresses, and the tangential stresses vanish all over the plane $z = 0$. The problem is to evaluate the normal stresses $\sigma(\rho, \phi)$ inside $\rho = a$. Many contact problems involving embedding of a cylindrical punch of an arbitrary profile in the half-space can be reduced to such a mixed boundary-value problem described here mathematically.

According to [5] the problem relates directly to the following integral equation given in polar coordinates

$$\int_0^a \int_0^{2\pi} \frac{\sigma(\rho_0, \phi_0)\rho_0 \, d\rho_0 \, d\phi_0}{[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)]^{(1+\alpha)/2}} = w(\rho, \phi) \text{ with } \begin{matrix} 0 \leq \rho \leq a, \\ 0 \leq \phi < 2\pi \end{matrix}. \tag{1}$$

For simplicity, the material elastic constants are incorporated within normal stresses, so the function σ in eqn (1) is given by the quotient of the stresses and the elastic constants. It is possible to reduce the integral equation (1) to a sequence of two Abel type integral operators and an L-operator to be introduced later. Inverse of the above mentioned operators can be found easily and thus the exact solution of (1) in closed form can be obtained. For this purpose it is necessary to obtain the integral representation of the kernel given by the following expressions:

$$\frac{1}{[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)]^{(1+\alpha)/2}} = \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\phi_0)}}{2\pi} \int_0^{2\pi} \frac{e^{-in\psi} \, d\psi}{[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\psi]^{(1+\alpha)/2}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\phi_0)}}{2\pi\rho_0^{1+\alpha}} \frac{2\pi\Gamma[|n| + (1+\alpha)/2]}{\Gamma[(1+\alpha)/2]\Gamma(|n|+1)} \left(\frac{\rho}{\rho_0}\right)^{|n|} {}_2F_1\left(\frac{1+\alpha}{2}, |n| + \frac{1+\alpha}{2}, |n|+1; \frac{\rho^2}{\rho_0^2}\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{2}{\pi} \cos\frac{\pi\alpha}{2} \frac{e^{in(\phi-\phi_0)}}{(\rho\rho_0)^{|n|}} \int_0^{\min(\rho,\rho_0)} \frac{x^{2|n|+\alpha}\, dx}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+\alpha)/2}}$$

$$= \frac{2}{\pi}\cos\frac{\pi\alpha}{2}\int_0^{\min(\rho,\rho_0)} \frac{x^\alpha\, dx}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+\alpha)/2}} \left[\frac{1}{1-\frac{x^2}{\rho\rho_0}e^{i(\phi-\phi_0)}} + \frac{1}{1-\frac{x^2}{\rho\rho_0}e^{-i(\phi-\phi_0)}} - 1\right] \quad (2)$$

Here, the Fourier series expansion and the hypergeometric functions properties described in [6] were employed. Substitution of the final expression (2) into (1) and further changing of the order of integration, will lead to the form

$$\frac{2}{\pi}\cos\frac{\pi\alpha}{2}\int_0^\rho \frac{x^\alpha\, dx}{(\rho^2-x^2)^{(1+\alpha)/2}}\int_x^a \frac{\rho_0\, d\rho_0}{(\rho_0^2-x^2)^{(1+\alpha)/2}}\, L\left(\frac{x^2}{\rho\rho_0}, \phi-\phi_0\right)\sigma(\rho_0,\phi_0) = w(\rho,\phi). \quad (3)$$

The scheme of changing of the order of integration used is explained by the identity

$$\int_0^\rho d\rho_0 \int_0^{\rho_0} dx + \int_\rho^a d\rho_0 \int_0^\rho dx = \int_0^\rho dx \int_x^\rho d\rho_0 + \int_0^\rho dx \int_\rho^a d\rho_0 = \int_0^\rho dx \int_x^a d\rho_0.$$

The newly introduced integral operator L may now be described by

$$L(k,\phi-\phi_0)f(\phi_0) = \int_0^{2\pi} [(1-k\, e^{i(\phi-\phi_0)})^{-1} + (1-k\, e^{-i(\phi-\phi_0)})^{-1} - 1] f(\phi_0)\, d\phi_0. \quad (4)$$

The following important properties of L-operators can be easily verified and are

$$L(k_1,\phi_1-\phi)\, L(k,\phi-\phi_0)f(\phi_0) = 2\pi L(kk_1,\phi_1-\phi_0)f(\phi_0)$$

$$L(1,\phi-\phi_0)f(\phi_0) = 2\pi f(\phi). \quad (5)$$

Using these properties, one can now construct the operator inverse of the L-operator as

$$L^{-1}(k,\phi-\phi_0) = \frac{1}{2\pi}L(k^{-1},\phi-\phi_0).$$

Now the integral equation (3) is equivalent to (1) and represents a sequence of two Abel type integral operators and an L-operator. As the inverse operators to all the three are known, the following procedure will lead to the exact solution of (1) in a closed form.

Application of the operator

$$\frac{d}{dr}\int_0^r \frac{\rho\, d\rho}{(r^2-\rho^2)^{(1-\alpha)/2}} L\left(\frac{\rho}{z}, \phi_1-\phi\right)$$

to the both parts of (3) yields

$$2\pi r^\alpha \int_r^a \frac{\rho_0\, d\rho_0}{(\rho_0^2-r^2)^{(1-\alpha)/2}} L\left(\frac{r^2}{z\rho_0}, \phi_1-\phi_0\right)\sigma(\rho_0,\phi_0) = \frac{d}{dr}\int_0^r \frac{\rho\, d\rho}{(r^2-\rho_0^2)^{(1-\alpha)/2}} L\left(\frac{\rho}{z}, \phi_1-\phi\right) w(\rho,\phi).$$

Both parts of the last expression may be divided by r^α, and applying the following operator

$$\frac{d}{dy}\int_y^a \frac{r\, dr}{(r^2-y^2)^{(1-\alpha)/2}} L\left(\frac{z^2}{r^2}, \phi_2-\phi_1\right).$$

One can then obtain the expression

$$-\frac{2\pi^3}{\cos\frac{\pi\alpha}{2}}\,yL\left(\frac{z}{y},\phi_2-\phi_0\right)\sigma(y,\phi_0)=\frac{d}{dy}\int_y^a\frac{r^{1-\alpha}\,dr}{(r^2-y^2)^{(1-\alpha)/2}}L\left(\frac{z^2}{r^2},\phi_2-\phi_1\right)$$

$$\times\frac{d}{dr}\int_0^r\frac{\rho\,d\rho}{(r^2-\rho^2)^{(1-\alpha)/2}}\,L\left(\frac{\rho}{z},\phi_1-\phi\right)w(\rho,\phi).$$

Now application of the operator $L(y/z,\psi-\phi_2)$ gives the type of solution sought for the contact problem and is

$$\sigma(y,\psi)=-\frac{\cos\frac{\pi\alpha}{2}}{8\pi^3y}L\left(\frac{y}{z},\psi-\phi_2\right)\frac{d}{dy}\int_y^a\frac{r^{1-\alpha}\,dr}{(r^2-y^2)^{(1-\alpha)/2}}L\left(\frac{z^2}{r^2},\phi_2-\phi_1\right)$$

$$\cdot\frac{d}{dr}\int_0^r\frac{\rho\,d\rho}{(r^2-\rho^2)^{(1-\alpha)/2}}\,L\left(\frac{\rho}{z},\phi_1-\phi\right)w(\rho,\phi). \tag{6}$$

The solution given in (6) can be simplified using the rules of differentiation under the integral sign and the L-operator properties states in (5). This yields

$$\sigma(y,\psi)=\frac{\cos\frac{\pi\alpha}{2}}{2\pi^3}\int_0^{2\pi}\left\{\frac{w(0,\phi)+\eta(a,y,\phi,\psi)}{(a^2-y^2)^{(1-\alpha)/2}}-\int_y^a\frac{dr}{(r^2-y^2)^{(1-\alpha)/2}}\frac{d}{dr}\eta(r,y,\phi,\psi)\right\}d\phi$$

$$\eta(r,y,\phi,\psi)=r^{1-\alpha}\int_0^r\frac{d\rho}{(r^2-\rho^2)^{(1-\alpha)/2}}\frac{d}{d\rho}L\left(\frac{y\rho}{r^2},\phi-\psi\right)w(\rho,\phi). \tag{7}$$

3. INVESTIGATION OF THE PROPERTIES OF THE SOLUTION

The evaluation of the resulting force P and the moments M_x and M_y acting on the punch as well as the normal displacements outside the contact region are of practical interest. It will be shown that all these parameters can be expressed in terms of the normal displacements inside $\rho=a$ by the following procedure.

Since

$$P=\int_0^a\int_0^{2\pi}\sigma(\rho,\phi)\rho\,d\rho\,d\phi, \tag{8}$$

substitution of (6) into (8) yields directly the resultant force

$$P=\frac{\cos\frac{\pi\alpha}{2}}{\pi^2}\int_0^a\int_0^{2\pi}\frac{w(\rho,\phi)\rho\,d\rho\,d\phi}{(a^2-\rho^2)^{(1-\alpha)/2}}. \tag{9}$$

For computation of the moments M_x and M_y it is convenient to introduce the complex parameter

$$M=M_x+iM_y=i\int_0^a\int_0^{2\pi}\sigma(\rho,\phi)\,e^{-i\phi}\rho^2\,d\rho\,d\phi \tag{10}$$

Since the first harmonics of σ will give the only one non-zero term after integration due to (10), then

$$M=-i\frac{\cos\frac{\pi\alpha}{2}}{\pi^2}\int_0^a\int_0^{2\pi}y^2\,dy\frac{d}{dy}\int_y^a\frac{r^{-1-\alpha}\,dr}{(r^2-y^2)^{(1-\alpha)/2}}\frac{d}{dr}\int_0^r\frac{w(\rho,\phi)\,e^{-i\phi}\rho^2\,d\rho\,d\phi}{(r^2-\rho^2)^{(1-\alpha)/2}}.$$

Integrating by parts and changing the order of integration gives the final expression for the resultant moment as

$$M = i\frac{2\cos\frac{\pi\alpha}{2}}{\pi^2(1+\alpha)}\int_0^a\int_0^{2\pi}\frac{w(\rho,\phi)\,e^{-i\phi}\rho^2\,d\rho\,d\phi}{(a^2-\rho^2)^{(1-\alpha)/2}}. \tag{11}$$

It may be noted that the expressions (9) and (11) correspond to the results previously reported in [5].

In reviewing the derivation of the expression (3) one may find that it is valid for the evaluation of the normal displacements outside the contact region if the upper limit of integration ρ is replaced by a. Now substitution of eqn (6) into adjusted form of eqn (3) results in

$$w(\rho,\phi) = \frac{\cos\frac{\pi\alpha}{2}}{\pi^2}\int_0^a\frac{dx}{(\rho^2-x^2)^{(1+\alpha)/2}}\frac{d}{dx}\int_0^x\frac{\rho_0\,d\rho_0}{(x^2-\rho_0^2)^{(1-\alpha)/2}}L\left(\frac{\rho_0}{\rho},\phi-\phi_0\right)w(\rho_0,\phi_0)$$

$$\text{for } \rho > a.$$

Performing the differentiation under the integral sign and then integrating with respect to x and by parts yields

$$w(\rho,\phi) = \frac{\cos\frac{\pi\alpha}{2}}{\pi^2}(\rho^2-a^2)^{(1-\alpha)/2}\int_0^a\int_0^{2\pi}\frac{w(\rho_0,\phi_0)\rho_0\,d\rho_0\,d\phi_0}{(a^2-\rho_0^2)^{(1-\alpha)/2}[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]}$$

$$\text{for } \rho > a. \tag{12}$$

Here the following identities were employed [6]

$$\frac{d}{dz}\left[z^{(1+\alpha)/2}{}_2F_1\left(\frac{1+\alpha}{2},\frac{1+\alpha}{2},\frac{3+\alpha}{2};z\right)\right] = \frac{1+\alpha}{2}z^{-(1-\alpha)/2}(1-z)^{-(1+\alpha)/2}$$

$$\frac{d}{dx}\int_0^x\frac{f(t)t\,dt}{(x^2-t^2)^{(1-\alpha)/2}} = f(0)x^\alpha + x\int_0^x\frac{df(t)}{(x^2-t^2)^{(1-\alpha)/2}}$$

Now the eqns (9) and (11) give the final expressions for the resulting force and moments acting upon the punch, and expression (12) gives the values of the normal displacements outside the contact region in terms of the known displacements inside. Expressions (9) and (12) correspond to the results given in [7] obtained for a homogeneous half-space.

4. ILLUSTRATIVE EXAMPLE

The problem is to determine the influence of a load outside of a smooth punch on the stresses under the punch. The boundary conditions, corresponding to the problem, are:

$$w = 0 \quad\text{and}\quad \tau = 0 \quad\text{for}\quad 0 \le \rho \le a,\quad 0 \le \phi < 2\pi$$

$$\sigma = \sigma(\rho,\phi) \quad\text{and}\quad \tau = 0 \quad\text{for}\quad a < \rho < \infty,\quad 0 \le \phi < 2\pi.$$

The normal stresses σ inside $\rho = a$ and the normal displacements w outside are to be determined as stated above.

For simplicity, consider a particular case of

$$\sigma(\rho,\phi) = P_0\delta(\rho-b)\delta(\phi-\phi_b)/\rho,$$

which corresponds to the action of a concentrated force P_0 applied at a point on the boundary with the polar coordinates $b, \phi_b (b > a)$. Here $\delta(\cdot)$ denotes the impulse delta-function. Using properties of δ-functions as well as eqn (3) one can obtain the following integral equation

$$\int_0^\rho \frac{x^\alpha \, dx}{(\rho^2 - x^2)^{(1+\alpha)/2}} \int_x^a \frac{\rho_0 \, d\rho_0}{(\rho_0^2 - x^2)^{(1+\alpha)/2}} L\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right) \sigma(\rho_0, \phi_0)$$

$$= -P_0 \int_0^\rho \frac{x^\alpha \, dx}{(\rho^2 - x^2)^{(1+\alpha)/2} (b^2 - x^2)^{(1+\alpha)/2}} \left[\frac{1}{1 - \frac{x^2}{\rho b} e^{i(\phi - \phi_b)}} + \frac{1}{1 - \frac{x^2}{\rho b} e^{-i(\phi - \phi_b)}} - 1 \right]$$

$$\text{for } \rho \le a. \tag{13}$$

Details of this derivation are presented in Appendix B.

Employing the same procedures as described in Section 2 before, the following solution for (13) can be derived to give the stress distribution and is

$$\sigma(\rho, \phi) = -\frac{P_0}{\pi^2} \cos\frac{\pi\alpha}{2} \left(\frac{b^2 - a^2}{a^2 - \rho^2}\right)^{(1-\alpha)/2} \frac{1}{\rho^2 + b^2 - 2b\rho \cos(\phi - \phi_b)} \quad \text{for } \rho \le a. \tag{14}$$

Returning to the original problem of an arbitrary pressure applied at the boundary at $\rho > a$ and making use of (14), the following integral gives the complete solution.

$$\sigma(\rho, \phi) = -\frac{\cos\frac{\pi\alpha}{2}}{\pi^2 (a^2 - \rho^2)^{(1-\alpha)/2}} \int_a^\infty \int_0^{2\pi} \frac{(\rho_0^2 - a^2)^{(1-\alpha)/2} \sigma(\rho_0, \phi_0) \rho_0 \, d\rho_0 \, d\phi_0}{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)}$$

$$\text{for } \rho \le a. \tag{15}$$

Expression (15) corresponds to the result obtained in a different context in [8].

The normal displacements of the boundary $z = 0$ outside the circle $\rho = a$ may then be evaluated as a superposition of the displacements caused by the applied pressure and those resulting from the stresses inside $\rho = a$, as determined by (15). Using a procedure analogous to the one described in Section 2, one can obtain the expression

$$w(\rho, \phi) = \frac{2}{\pi} \cos\frac{\pi\alpha}{2} \left\{ \int_\rho^\infty \frac{x^\alpha \, dx}{(x^2 - \rho^2)^{(1+\alpha)/2}} \int_a^x \frac{y \, dy}{(x^2 - y^2)^{(1+\alpha)/2}} L\left(\frac{\rho y}{x^2}, \phi - \psi\right) \sigma(y, \psi) \right.$$

$$\left. + \int_0^a \frac{x^\alpha \, dx}{(\rho^2 - x^2)^{(1+\alpha)/2}} \int_x^a \frac{\rho_0 \, d\rho_0}{(\rho_0^2 - x^2)^{(1+\alpha)/2}} L\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right) \sigma(\rho_0, \phi_0) \right\}$$

$$\text{for } \rho > a. \tag{16}$$

Substitution of (15) into the second integral of (16) leads, after simplification, to

$$w(\rho, \phi) = \frac{2}{\pi} \cos\frac{\pi\alpha}{2} \int_a^\rho \frac{x^\alpha \, dx}{(\rho^2 - x^2)^{(1+\alpha)/2}} \int_x^\infty \frac{\rho_0 \, d\rho_0}{(\rho_0^2 - x^2)^{(1+\alpha)/2}} L\left(\frac{x^2}{\rho\rho_0}, \phi - \phi_0\right) \sigma(\rho_0, \phi_0),$$

$$\text{for } \rho > a. \tag{17}$$

The details of this derivation are presented in the Appendix A. The expression (15) determines the stresses inside the circle $\rho = a$ and the normal displacements of the boundary outside $\rho = a$ are given by (17) in terms of the applied pressure.

As a simple illustration of the results obtained, a specific problem may now be considered. Let $\sigma(\rho, \phi) = \sigma_0 = \text{const}$ at the annulus $b \le \rho \le c$ $(b > a)$ and $\sigma(\rho, \phi) = 0$ for $a < b$ and $\rho > c$.

For this particular case, using (15) and (17), the solutions are derived as

$$\sigma(\rho,\phi) = -\frac{2\sigma_0\cos\frac{\pi\alpha}{2}}{\pi(a^2-\rho^2)^{(1-\alpha)/2}}\int_b^c \frac{(\rho_0^2-a^2)^{(1-\alpha)/2}\rho_0\,d\rho_0}{\rho_0^2-\rho^2} \quad \text{for } \rho \le a \tag{18}$$

$$w(\rho,\phi) = \frac{4\cos\frac{\pi\alpha}{2}}{1-\alpha}\left[\int_a^{\min(\rho,c)} \frac{(c^2-x^2)^{(1-\alpha)/2}}{(\rho^2-x^2)^{(1+\alpha)/2}}x^\alpha\,dx - \int_a^{\min(\rho,b)} \frac{(b^2-x^2)^{(1-\alpha)/2}}{(\rho^2-x^2)^{(1+\alpha)/2}}x^\alpha\,dx\right]$$

$$\text{for } \rho > a. \tag{19}$$

In general, the integrals (18) and (19) can be evaluated in terms of hypergeometric functions. In any particular case of an isotropic body ($\alpha = 0$), the integral (18) may be expressed in terms of elementary functions

$$\sigma(\rho,\phi) = -\frac{2}{\pi}\sigma_0\left[\frac{\sqrt{(c^2-a^2)}-\sqrt{(b^2-a^2)}}{\sqrt{(a^2-\rho^2)}} - \tan^{-1}\sqrt{\left(\frac{c^2-a^2}{a^2-\rho^2}\right)} + \tan^{-1}\sqrt{\left(\frac{b^2-a^2}{a^2-\rho^2}\right)}\right].$$

5. DISCUSSION

Some of the results presented may be considered as of general mathematical interest in many mechanics problems. For example, on the basis of (12), the following theorem may be established. That is, if a function $w(\rho,\phi)$ has the integral representation (1) inside a circle $\rho = a$, and the density $\sigma = 0$ outside, then it is possible to evaluate the values of the function $w(\rho,\phi)$ outside the circle directly through its values inside, by using (12) without the determination of the density σ.

The second theorem may be established on the basis of the expression (15). If a function $\sigma(\rho,\phi)$ serves as a density in the integral representation, outside $\rho = a$,

$$w(\rho,\phi) = \int_0^\infty\int_0^{2\pi} \frac{\sigma(\rho_0,\phi_0)\rho_0\,d\rho_0\,d\rho_0}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)]^{(1+\alpha)/2}} \quad \text{for } \rho > a$$

and the function $w(\rho,\phi) = 0$ inside $\rho = a$, then it is possible to evaluate the values of σ inside the circle directly through its values outside using (15) derived.

Both these theorems establish a certain mathematical similarity between such physically different parameters as stresses and displacements.

Another general relationship can be derived by multiplication of both parts of eqn (3) by $\rho^{|n|}e^{in\phi}(a^2-\rho^2)^{-(1-\alpha)/2}$ and integrating with respect to ρ and ϕ over the circle $\rho \le a$. The result is then expressed as

$$\int_0^{2\pi}\int_0^a \sigma(\rho,\phi)\rho^{1+|n|}e^{in\phi}\,d\rho\,d\phi = \frac{\Gamma(1+|n|)\left(\frac{1+\alpha}{2}\right)}{\pi^2\Gamma\left(|n|+\frac{1+\alpha}{2}\right)}\cos\frac{\pi\alpha}{2}$$

$$\times\int_0^{2\pi}\int_0^a \frac{w(\rho,\phi)\rho^{1+|n|}e^{in\phi}\,d\rho\,d\phi}{(a^2-\rho^2)^{(1-\alpha)/2}}.$$

One can easily notice that the expression defining the resulting force (9) and the one defining the resulting moment (11) are particular cases of the previous relationship for $n = 0$ and $n = -1$ respectively.

REFERENCES

1. N. A. Rostovtsev and I. E. Khranevskaya, The solution of the Boussinesque problem for a half-space whose modulus of elasticity is a power function of depth. *PMM* 35, 1000 (1971).
2. V. P. Plevako, Action of shearing forces applied to the boundary of non-homogeneous half-space. *Sov. Appl. Mech.* 9(11), 49 (1973).

3. A. E. Puro, Application of Hankel transforms to the solution of axisymmetric problems when the modulus of elasticity is a power function of depth. *PMM* 37, 896 (1973).
4. G. Ya. Popov, Axisymmetric contact problems for an elastic inhomogeneous half-space in the presence of cohesion. *PMM* 37, 1052 (1973).
5. N. A. Rostovtsev, On certain solutions of an integral equation of the theory of a linearly deformable foundation. *PMM* 28, 127 (1964).
6. A. Erdelyi *et al.*, *Higher Transcendental Functions*, Vol. 1. McGraw-Hill, New York (1954).
7. J. T. Guidera, R. W. Lardner and G. E. Tupholme, The indentation of a half-space of hexagonal elastic material by a circular punch of arbitrary end-profile. *J. Engng Math.* 12, 77 (1978).
8. V. I. Mossakovski and L. R. Mossakovskaya, Influence on the contact pressure at the base of a punch of a load acting away from the punch. *Prikl. Mekh.* 1, 132 (1965). (in Russian).

APPENDIX A

The derivation of expression (17) in the paper is given here. For simplicity all the intermediate transformations are performed for nth harmonics of the stresses and the displacements and the final result is obtained by summation. By eqn (16), the following expression is valid for the nth harmonics of the displacements:

$$w_n(\rho)=4\cos\frac{\pi\alpha}{2}\left\{\rho^n\int_\rho^\infty\frac{x^{\alpha-2n}\,dx}{(x^2-\rho^2)^{(1+\alpha)/2}}\int_a^x\frac{\rho_0^{1+n}\sigma_n(\rho_0)\,d\rho_0}{(x^2-\rho_0^2)^{(1+\alpha)/2}}+\frac{1}{\rho^n}\int_0^a\frac{x^{\alpha+2n}\,dx}{(\rho^2-x^2)^{(1+\alpha)/2}}\int_x^a\frac{y^{1-n}\sigma_n(y)\,dy}{(y^2-x^2)^{(1+\alpha)/2}}\right\},\quad \text{for } \rho>a. \tag{A1}$$

According to (15), $\sigma_n(y)$ in the second integral of (A1) may be expressed in terms of $\sigma_n(\rho_0)$ as

$$\sigma_n(y)=-\frac{2y^n\cos\frac{\pi\alpha}{2}}{\pi(a^2-y^2)^{(1-\alpha)/2}}\int_a^\infty\frac{(\rho_0^2-a^2)^{(1-\alpha)/2}\rho_0^{1-n}\sigma_n(\rho_0)\,d\rho_0}{\rho_0^2-y^2} \tag{A2}$$

Substitution of (A2) into (A1) and integration with respect to y, yields

$$w_n(\rho)=4\cos\frac{\pi\alpha}{2}\left\{\rho^n\int_\rho^\infty\frac{x^{\alpha-2n}\,dx}{(x^2-\rho^2)^{(1+\alpha)/2}}\int_a^x\frac{\rho_0^{1+n}\sigma_n(\rho_0)\,d\rho_0}{(x^2-\rho_0^2)^{(1+\alpha)/2}}-\frac{1}{\rho^n}\int_0^a\frac{x^{\alpha+2n}\,dx}{(\rho^2-x^2)^{(1+\alpha)/2}}\int_a^\infty\frac{\sigma_n(\rho_0)\rho_0^{1-n}\,d\rho_0}{(\rho_0^2-x^2)^{(1+\alpha)/2}}\right\},\quad \text{for } \rho>a \tag{A3}$$

Changing the order of integration of the first term leads to

$$\rho^n\int_\rho^\infty\frac{x^{\alpha-2n}\,dx}{(x^2-\rho^2)^{(1+\alpha)/2}}\int_a^x\frac{\rho_0^{1+n}\sigma_n(\rho_0)\,d\rho_0}{(x^2-\rho_0^2)^{(1+\alpha)/2}}=\rho^n\int_a^\rho\rho_0^{1+n}\sigma_n(\rho_0)\,d\rho_0\int_\rho^\infty\frac{x^{\alpha-2n}\,dx}{[(x^2-\rho^2)(x^2-\rho_0^2)]^{(1+\alpha)/2}}$$
$$+\rho^n\int_\rho^\infty\rho_0^{1+n}\sigma_n(\rho_0)\,d\rho_0\int_{\rho_0}^\infty\frac{x^{\alpha-2n}\,dx}{[(x^2-\rho^2)(x^2-\rho_0^2)]^{(1+\alpha)/2}}.$$

Making use of identities

$$\int_{\rho_0,\rho}^\infty\frac{x^{\alpha-2n}\,dx}{[(x^2-\rho^2)(x^2-\rho_0^2)]^{(1+\alpha)/2}}=\frac{1}{(\rho\rho_0)^{2n}}\int_0^{\rho,\rho_0}\frac{x^{\alpha+2n}\,dx}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+\alpha)/2}}$$

and substituting the last expression into the previous one results in

$$\rho^n\int_\rho^\infty\frac{x^{\alpha-2n}\,dx}{(x^2-\rho^2)^{(1+\alpha)/2}}\int_a^x\frac{\rho_0^{1+n}\sigma_n(\rho_0)\,d\rho_0}{(x^2-\rho_0^2)^{(1+\alpha)/2}}=\frac{1}{\rho^n}\int_a^\rho\rho_0^{1-n}\sigma_n(\rho_0)\,d\rho_0\int_0^{\rho_0}\frac{x^{\alpha+2n}\,dx}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+\alpha)/2}}$$
$$+\frac{1}{\rho^n}\int_\rho^\infty\rho_0^{1-n}\sigma_n(\rho_0)\,d\rho_0\int_0^\rho\frac{x^{\alpha+2n}\,dx}{[(\rho^2-x^2)(\rho_0^2-x^2)]^{(1+\alpha)/2}}.$$

Changing the order of integration by the scheme

$$\int_0^\rho d\rho_0\int_0^\rho dx+\int_0^\infty d\rho_0\int_0^\rho dx=\int_a^\rho dx\int_x^\infty d\rho_0+\int_0^a dx\int_a^\infty d\rho_0$$

and substituting the result in (A3) leads, after simplification, to the form

$$w_n(\rho)=\frac{4}{\rho^n}\cos\frac{\pi\alpha}{2}\int_a^\rho\frac{x^{\alpha+2n}\,dx}{(\rho^2-x^2)^{(1+\alpha)/2}}\int_x^\infty\frac{\sigma_n(\rho_0)\rho_0^{1-n}\,d\rho_0}{(\rho_0^2-x^2)^{(1+\alpha)/2}}. \tag{A4}$$

Since $w(\rho,\phi)=\sum_{n=-\infty}^{\infty}w_n(\rho)\,e^{in\phi}$, summation of (A4) gives the final result (17).

APPENDIX B

The details of the derivation of eqn (13) are presented here. If arbitrary normal stresses $\sigma(\rho, \phi)$ are prescribed inside a circle $\rho = c$, then the normal displacements, produced by those stresses, are given by (3) as

$$w(\rho, \phi) = \frac{2}{\pi} \cos \frac{\pi \alpha}{2} \int_0^{\rho} \frac{x^{\alpha}\, dx}{(\rho^2 - x^2)^{(1+\alpha)/2}} \int_x^c \frac{\rho_0\, d\rho_0}{(\rho_0^2 - x^2)^{(1+\alpha)/2}} L\left(\frac{x^2}{\rho \rho_0}, \phi - \phi_0\right) \sigma(\rho_0, \phi_0) \tag{B1}$$

In the particular case of a normal concentrated force P_0, applied at a point with the polar coordinates b, $\phi_b (c > b > a)$, the given stresses may then be presented as

$$\sigma(\rho, \phi) = P_0 \delta(\rho - b) \delta(\phi - \phi_b)/\rho. \tag{B2}$$

Here $\delta(\cdot)$ denotes the impulse delta-function. The substitution of (B2) into (B1) results in

$$w(\rho, \phi) = \frac{2P_0}{\pi} \cos \frac{\pi \alpha}{2} \int_0^{\rho} \frac{x^{\alpha}\, dx}{[(\rho^2 - x^2)(b^2 - x^2)]^{(1+\alpha)/2}} \left[\left(1 - \frac{x^2}{\rho b} e^{i(\phi - \phi_b)}\right)^{-1} + \left(1 - \frac{x^2}{\rho b} e^{-i(\phi - \phi_b)}\right)^{-1} - 1\right]. \tag{B3}$$

Here the integral representation of the L-operator (4) and the following fundamental property of the delta-function were employed.

$$\int_a^b f(x) \delta(x - y)\, dx = f(y), \quad (a \le y \le b) \tag{B4}$$

In the example, considered in Section 4, the normal displacements inside the circle $\rho = a$ should be zero, therefore, the normal stresses inside the circle should provide the normal displacements equal and opposite to the ones defined by (B3). Now substitution of (G3) with opposite sign into (3) immediately gives (13).